Regularity of rational vertex operator algebras

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1 Introduction

Rational vertex operator algebras, which play a fundamental role in rational conformal field theory (see [BPZ] and [MS]), single out an important class of vertex operator algebras. Most vertex operator algebras which have been studied so far are rational vertex operator algebras. Familiar examples include the moonshine module $V^\natural$ ([B], [FLM], [D2]), the vertex operator algebras V_L associated with positive definite even lattices L ([B], [FLM], [D1]), the vertex operator algebras $L(l,0)$ associated with integrable representations of affine Lie algebras [FZ] and the vertex operator algebras $L(c_{p,q},0)$ associated with irreducible highest weight representations for the discrete series of the Virasoro algebra ([DMZ] and [W]).

A rational vertex operator algebra as studied in this paper is a vertex operator algebra such that any *admissible* module is a direct sum of simple ordinary modules (see Section 2). It is natural to ask if such complete reducibility holds for an arbitrary weak module (defined in Section 2). A rational vertex operator algebra with this property is called a *regular* vertex operator algebra. One motivation for studying such vertex operator algebras arises in trying to understand the appearance of negative fusion rules (which are computed by the Verlinde formula) for vertex operator algebras $L(l,0)$ for certain rational l (cf. [KS] and [MW]).

In this paper we give several sufficient conditions under which a rational vertex operator algebra is regular. We prove that the rational vertex operator algebras $V^\natural$, $L(l,0)$ for positive integers l, $L(c_{p,q},0)$ and V_L for positive definite even lattices L are regular. Our result for $L(l,0)$ implies that any restricted integrable module of level l for the corresponding affine Lie algebra is a direct sum of irreducible highest weight integrable modules. This result is expected to be useful in comparing the construction of tensor product of modules for $L(l,0)$ in [F] based on Kazhdan-Lusztig's approach [KL] with the construction of tensor product of modules [HL] in this special case. We should remark that V_L in general is a vertex algebra in the sense of [DL] if L is not positive definite. In this case we establish the complete reducibility of any weak module.

Since the definition of vertex operator algebra is by now well-known, we do not define vertex operator algebra in this paper. We refer the reader to [FLM] and [FHL] for their elementary properties. The reader can find the details of the constructions of $V^\natural$ and V_L in [FLM], and $L(l,0)$ and $L(c_{p,q},0)$ in [DMZ], [DL], [FLM], [FZ], [L1] and [W].

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The paper is organized as follows: In Section 2, after defining the notion of weak module for a vertex operator algebra and the definition of rational vertex operator algebra, we discuss the rational vertex operator algebras $V^\natural$, V_L, $L(l,0)$ and $L(c_{p,q},0)$. Section 3 is devoted to regular vertex operator algebras. We begin this section with the definition of regular vertex operator algebra. We show that the tensor product of regular vertex operator algebras is also regular and that a rational vertex operator algebra is regular under either of the assumptions (i) it contains a regular vertex operator subalgebra, or (ii) any weak module contains a simple ordinary module. These results are then used to prove that $V^\natural$, V_L (L is positive definite), $L(l,0)$ and $L(c_{p,q},0)$ are regular. We also discuss the complete reducibility of weak V_L-modules for an arbitrary even lattice L. Based on these results, we conjecture that *any* rational vertex operator algebra is regular.

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2 Rational vertex operator algebras

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra (cf. [B], [FHL] and [FLM]). A *weak module* M for V is a vector space equipped with a linear map

$$\begin{aligned}&V \to (\mathrm{End}\, M)[[z^{-1}, z]]\\ &v \mapsto Y_M(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \mathrm{End}\, M)\end{aligned}$$

(where for any vector space W, we define $W[[z^{-1}, z]]$ to be the vector space of W-valued formal series in z) satisfying the following conditions for $u, v \in V$, $w \in M$:

$$Y_M(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \quad \text{for} \quad v \in V; \tag{2.1}$$

$$v_n w = 0 \quad \text{for} \quad n \in \mathbb{Z} \;\; \text{sufficiently large}; \tag{2.2}$$

$$Y_M(\mathbf{1}, z) = 1; \tag{2.3}$$

$$\begin{aligned}z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) Y_M(u,z_1)Y_M(v,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right) Y_M(v,z_2)Y_M(u,z_1)\\ = z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) Y_M(Y(u,z_0)v, z_2).\end{aligned} \tag{2.4}$$

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3-m)\delta_{m+n,0}(\mathrm{rank}\, V) \tag{2.5}$$

for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \quad \text{for} \quad n \in \mathbb{Z}, \quad \text{i.e.,} \quad Y_M(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2};$$

$$\frac{d}{dz} Y_M(v,z) = Y_M(L(-1)v, z). \tag{2.6}$$

This completes the definition. We denote this module by (M, Y_M) (or briefly by M).

Definition 2.1 *An (ordinary) V-module is a weak V-module which carries a $\mathbb{C}$-grading*

$$M = \coprod_{\lambda \in \mathbb{C}} M_\lambda$$

such that $\dim M_\lambda$ *is finite and* $M_{\lambda+n} = 0$ *for fixed* λ *and* $n \in \mathbb{Z}$ *small enough. Moreover one requires that* M_λ *is the* λ*-eigenspace for* $L(0)$:

$$L(0)w = \lambda w = (wt\, w)w, \quad w \in M_\lambda.$$

This definition is weaker than that of [FLM], for example, where the grading on M is taken to be rational. The extra flexibility attained by allowing $\mathbb{C}$-gradings is important − see for example [DLM1] and [Z].

We observe some redundancy in the definition of weak module:

Lemma 2.2 *Relations (2.5) and (2.6) in the definition of weak module are consequences of (2.1)-(2.4).*

Proof. To establish (2.6) note that $L(-1)u = L(-1)u_{-1}\mathbf{1} = u_{-2}\mathbf{1}$ for $u \in V$. Then

$$\begin{aligned}
&Y_M(L(-1)u, z_2) \\
&= Y_M(u_{-2}\mathbf{1}, z_2) \\
&= \mathrm{Res}_{z_0} z_0^{-2} Y_M(Y(u, z_0)\mathbf{1}, z_2) \\
&= \mathrm{Res}_{z_0}\mathrm{Res}_{z_1} z_0^{-2}\left(z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(u, z_1)Y_M(\mathbf{1}, z_2)\right. \\
&\qquad \left. - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(\mathbf{1}, z_2)Y_M(u, z_1)\right) \\
&= \mathrm{Res}_{z_0}\mathrm{Res}_{z_1} z_0^{-2} z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(u, z_1) \\
&= \mathrm{Res}_{z_0}\mathrm{Res}_{z_1} z_0^{-2} z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) Y_M(u, z_2 + z_0) \\
&= \mathrm{Res}_{z_0} z_0^{-2} Y_M(u, z_2 + z_0) \\
&= \mathrm{Res}_{z_0} z_0^{-2} e^{z_0 \frac{d}{dz_2}} Y_M(u, z_2) \\
&= \frac{d}{dz_2} Y_M(u, z_2).
\end{aligned} \tag{2.7}$$

This establishes (2.6), and together with (2.4) and $Y(\omega, z_0)\omega = \frac{1}{2}(\mathrm{rank}V)z_0^{-4} + 2\omega z_0^{-2} + L(-1)\omega z_0^{-1} +$ regular terms we can easily deduce (2.5). □

Thus we may just use (2.1)-(2.4) as the axioms for a weak V-module.

Definition 2.3 *An* admissible *V-module is a weak V-module M which carries a $\mathbb{Z}_+$-grading*

$$M = \coprod_{n \in \mathbb{Z}_+} M(n)$$

($\mathbb{Z}_+$ is the set all nonnegative integers) satisfying the following condition: if $r, m \in \mathbb{Z}, n \in \mathbb{Z}_+$ and $a \in V_r$ then

$$a_m M(n) \subseteq M(r+n-m-1). \tag{2.8}$$

We call an admissible V-module M simple *in case 0 and M are the only $\mathbb{Z}_+$-graded submodules.*

V is called rational *if every admissible V-module is a direct sum of simple admissible V-modules. That is, we have complete reducibility of admissible V-modules.*

Remark 2.4 *(i) Note that any ordinary V-module is admissible.*

(ii) It is proved in [DLM1] that if V is rational then conversely, every simple admissible V-module is an ordinary module. Moreover V has only a finite number of inequivalent simple modules.

(iii) Zhu's definition of rational vertex operator algebra V is as follows [Z]: (a) all admissible V-module are completely reducible, (b) each simple admissible V-module is an ordinary V-module, (c) V only has finitely many inequivalent simple modules. Thanks to (ii), Zhu's definition of rational thus coincides with our own.

We next introduce a certain category $\mathcal{O}$ of admissible V-modules in analogy with the well-known category $\mathcal{O}$ of Bernstein-Gelfand-Gelfand. First some notation: for any weak V-module M we set for $h \in \mathbb{C}$:

$$M_h = \{m \in M | (L(0)-h)^k m = 0 \text{ for some } k \in \mathbb{Z}_+\}.$$

So M_h is a *generalized* eigenspace for $L(0)$, and in particular M_h is the h-eigenspace for $L(0)$ if $L(0)$ is a semisimple operator.

Now define $\mathcal{O}$ to be the category of weak V-modules M satisfying the following two conditions:

(1) $L(0)$ is locally finite in the sense that if $m \in M$ then there is a finite-dimensional $L(0)$-stable subspace of M which contains m.

(2) There are $h_1, \cdots, h_k \in \mathbb{C}$ such that

$$M = \oplus_{i=1}^k \oplus_{n \in \mathbb{Z}_+} M_{n+h_i}.$$

These are the *objects* of $\mathcal{O}$. Morphisms may be taken to be V-module homomorphisms, though we will not make use of them in the sequel.

Remark 2.5 *(i) Any weak V-module which belongs to $\mathcal{O}$ is necessarily admissible: a $\mathbb{Z}_+$-grading obtains by defining $M(n) = \oplus_{i=1}^k M_{h_i+n}$. Condition (2.8) follows in the usual way.*

(ii) Suppose that M is a weak V-module and that W is a weak V-submodule of M. Then M lies in $\mathcal{O}$ if, and only if, both W and M/W lie in $\mathcal{O}$.

(iii) If V is rational, any weak V-module in $\mathcal{O}$ is a direct sum of simple V-modules (use Remark 2.4 (ii)).

Next we briefly discuss some familiar examples of rational vertex operator algebras. The reader is referred to the references for notation and the details of the constructions.

(1) Let L be an even lattice and V_L the corresponding vertex algebra (see [B], [DL] and [FLM]). It is proved in [D1] that if L is positive definite then V_L is rational and its simple modules are parametrized by L'/L where L' is the dual lattice of L.

(2) Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra with a Cartan subalgebra $\mathfrak{h}$ and $\hat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}c$ the corresponding affine Lie algebra. Fix a positive integer l. Then any $\lambda \in \mathfrak{h}^*$ can be viewed as a linear form on $\mathbb{C}c \oplus \mathfrak{h} \subset \hat{\mathfrak{g}}$ by sending c to l. Let us denote the corresponding irreducible highest weight module for $\hat{\mathfrak{g}}$ by $L(l, \lambda)$. Then $L(\ell, 0)$ is a rational vertex operator algebra ([DL], [FZ], [L1]).

(3) Let c and h be two complex numbers and let $L(c, h)$ be the lowest weight irreducible module for the Virasoro algebra with central charge c and lowest weight h. Then $L(c, 0)$ has a natural vertex operator algebra structure (cf. [FZ]). Moreover, $L(\ell, 0)$ is rational if, and only if, $c = c_{p,q} = 1 - \frac{6(p-q)^2}{pq}$ for $p, q \in \{2, 3, 4, \cdots\}$ and p, q are relatively prime (see [DMZ] and [W]).

(4) Let $V^\natural$ be the moonshine module vertex operator algebra constructed by Frenkel, Meurman and Lepowsky [FLM] (see also [B]). It is established in [D2] that $V^\natural$ is *holomorphic* in the sense that $V^\natural$ is rational and the only simple module is $V^\natural$ itself.

(5) Let $V^1, \cdots, V^k$ be vertex operator algebras. Then $V = \otimes_{i=1}^k V^i$ is a vertex operator algebra of rank $\sum_{i=1}^k \mathrm{rank} V^i$ and any simple V-module M is isomorphic to a tensor product module $M^1 \otimes \cdots \otimes M^k$ for some simple V^i-module M^i [FHL]. Furthermore, $V^1 \otimes V^2 \otimes \cdots \otimes V^k$ is rational if, and only if, each V^i is rational [DMZ].

(6) Let $V^1, \cdots, V^k$ be vertex operator algebras of the same rank. Then $\oplus_{i=1}^k V^k$ is a vertex operator algebra [FHL]. It is clear that $\oplus_{i=1}^k V^i$ is rational if each V^i is rational. The vacuum space of the resulting vertex operator algebra is not one-dimensional, and we will not consider this particular example further in this paper.

3 Regular vertex operator algebras

In Section 2 we made use of the complete reducibility of admissible modules in order to define rational vertex operator algebras. The study of complete reducibility of an arbitrary weak module for a rational vertex operator algebra leads us to the following notion of regular vertex operator algebra:

Definition 3.1 *A vertex operator algebra V is said to be* regular *if any weak V-module M is a direct sum of simple ordinary V-modules.*

Remark 3.2 *A regular vertex operator algebra V is necessarily rational. Indeed if M is a weak V-module then, being a direct sum of ordinary simple V-module, it is admissible (Remark 2.4 (i)) and, for the same reason, a direct sum of simple admissibles.*

The main result of the present paper is to show that the rational vertex operator algebras in examples (1)-(4) of Section 2 are each regular. First we have some general results which will be useful later.

Proposition 3.3 *Let $V^1, \cdots, V^k$ be regular vertex operator algebras. Then $V = V^1 \otimes V^2 \otimes \cdots \otimes V^k$ is regular.*

Proof. Let M be a weak V-module. For each $1 \leq i \leq k$, we may regard V^i as a vertex operator subalgebra with a different Virasoro element. Then M is a weak V^i-module. Since V^i is regular, M is a direct sum of simple ordinary V^i-modules. Note that there are only finitely many simple V^i-modules up to equivalence. Thus M is a V^i-module in category $\mathcal{O}$ of weak V^i-modules. Denote the generators of the Virasoro algebra of V^i by $L_i(n)$. Then $L(0) = L_1(0) + \cdots + L_k(0)$ and $L_i(0)$'s commute with each other. This implies that M is in category $\mathcal{O}$ of weak V-modules. So M is completely reducible by Remarks 3.2 and 2.5 (iii). $\square$

Proposition 3.4 *Let V be a rational vertex operator algebra such that there is a regular vertex operator subalgebra U with the same Virasoro element ω. Then V is regular.*

Proof. Let M be a weak V-module. Then M is a weak U-module, so that M is a direct sum of simple ordinary U-modules. Thus $L(0)$ acts semisimply on M. Let $W^1, \cdots, W^k$ be all simple U-modules up to equivalence. Then we can write $M = \sum_{i=1}^{k} \oplus_{n\in\mathbb{Z}_+} M_{n+h_i}$, where h_i is the lowest weight of W^i. Thus M is in the category $\mathcal{O}$ for V. The complete reducibility of M follows immediately as V is rational. $\square$

Proposition 3.5 *Let V be a rational vertex operator algebra such that any nonzero weak V-module contains a simple ordinary V-submodule. Then V is regular.*

Proof. Let M be any weak V-module and let W be the sum of all simple ordinary submodules. We have to prove that $W = M$. If $M \neq W$, M/W is a nonzero weak V-module so that by assumption, there is a simple ordinary V-submodule M^1/W of M/W. Then both W and M^1/W are in the category $\mathcal{O}$. Thus M^1 is in the category $\mathcal{O}$ and M^1 is a direct sum of simple ordinary V-modules. This contradicts the choice of W. $\square$

Now we are ready to show that the vertex operator algebras $L(l,0)$, $L(c_{p,q},0)$, $V^\natural$ and V_L are regular.

Recall from example (2) that $\mathfrak{g}$ is a finite-dimensional simple Lie algebra with Cartan subalgebra $\mathfrak{h}$; $L(l,0)$ is a vertex operator algebra. We shall denote the corresponding root system by Δ.

Lemma 3.6 *There is a basis $\{a^1, \cdots, a^m\}$ for $\mathfrak{g}$ such that for $1 \leq i,j \leq m$ we have*

$$[Y(a^i,z_1), Y(a^i,z_2)] = 0 \quad \textit{and } Y(a^i,z)^{3\ell+1} = 0 \tag{3.1}$$

as operators on $L(\ell,0)$.

Proof. Let $\alpha \in \Delta$ and $e \in \mathfrak{g}_\alpha$. If $\mathfrak{g}$ is of type A, D, or E, it is proved in [MP1] and [DL] that $Y(e,z)^{\ell+1} = 0$. In general, it is proved in [L1] and [MP2] that $Y(e,z)^{3\ell+1} = 0$. It is well known (cf. [H], [K1]) that there are elements $e_\alpha \in \mathfrak{g}_\alpha, f_\alpha \in \mathfrak{g}_{-\alpha}, h_\alpha \in \mathfrak{h}$ which linearly span a subalgebra naturally isomorphic to sl_2. Set $\sigma_\alpha = e^{(e_\alpha)_0}$ where $(e_\alpha)_0$ is the component operator of $Y(e_\alpha, z)$ (cf. (2.1)) corresponding to z^{-1}. Then σ_α is an automorphism of the vertex operator algebra $L(\ell,0)$ (see Chap. 11 of [FLM]). A straightforward calculation gives $\sigma_\alpha(f_\alpha) = f_\alpha + h_\alpha - 2e_\alpha$. Since $e_\alpha, f_\alpha, h_\alpha$ form a basis of $\mathfrak{g}$ for $\alpha \in \Delta$, $e_\alpha, f_\alpha, \sigma_\alpha(f_\alpha)$ also form a basis of $\mathfrak{g}$. It is clear that this basis satisfies condition (3.1). $\square$

Theorem 3.7 *Let ℓ be a positive integer. Then the vertex operator algebra $L(\ell,0)$ is regular.*

Proof. By Proposition 3.5, it is enough to prove that any nonzero weak $L(\ell,0)$-module M contains a simple $L(\ell,0)$-module. This will be established in three steps.

Claim 1: *There exists a nonzero $u \in M$ such that $(t\mathbb{C}[t] \otimes \mathfrak{g})u = 0$.* Set $\mathfrak{g}(n) = t^n \otimes \mathfrak{g}$ for $n \neq 0$. For any nonzero $u \in M$, by the definition of a weak module, $\mathfrak{g}(n)u = 0$ for sufficiently large n. So $(t\mathbb{C}[t] \otimes \mathfrak{g})u$ is finite-dimensional. For any $u \in M$, we define $d(u) = \dim(t\mathbb{C}[t] \otimes \mathfrak{g})u$. If there is a $0 \neq u \in M$ such that $d(u) = 0$, then $(t\mathbb{C}[t] \otimes \mathfrak{g})u = 0$. Suppose that $d(u) > 0$ for any $0 \neq u \in M$. Take $0 \neq u \in M$ such that $d(u)$ is minimal.

Let a^i $(1 \leq i \leq m)$ be a basis of $\mathfrak{g}$ satisfying condition (3.1) and let k be the positive integer such that $\mathfrak{g}(k)u \neq 0$ and $\mathfrak{g}(n)u = 0$ whenever $n > k$. By definition of k, $a^i(k)u \neq 0$ for some $1 \leq i \leq m$. Since $Y(a^i,z)^{3\ell+1} = 0$, by Proposition 13.16 in [DL], $Y_M(a^i,z)^{3\ell+1} = 0$. Extracting the coefficient of $z^{-(k+1)(3\ell+1)}$ from $Y_M(a^i,z)^{3\ell+1}u = 0$ we obtain $(a^i_k)^{3\ell+1}u = 0$.

Let r be a nonnegative integer such that $(a^i_k)^r u \neq 0$ and $(a^i_k)^{r+1}u = 0$. Set $v = (a^i_k)^r u$. We will obtain a contradiction by showing that $d(v) < d(u)$. First we prove that if $a_n u = 0$ for some $a \in \mathfrak{g}, 1 \leq n \in \mathbb{Z}$, then $a_n v = 0$. In the following we will show by induction on m that $a_n(a^i_k)^m u = 0$ for any $a \in \mathfrak{g}$ and $m \in \mathbb{Z}$ nonnegative. If $m = 0$ this is immediate by the choice of u. Now assume that the result holds for m. Since $[a,a^i]_{k+n}u = 0$ (from the definition of k) and $a_n u = 0$, by the induction assumption that $a_n(a^i_k)^m u = 0$ we have:

$$[a,a^i]_{k+n}(a^i_k)^m u = 0,\ a_n(a^i_k)^m u = 0. \tag{3.2}$$

Thus

$$\begin{aligned} a_n(a^i_k)^{m+1}u &= [a_n, a^i_k](a^i_k)^m u + a^i_k a_n (a^i_k)^m u \\ &= [a,a^i]_{k+n}(a^i_k)^m u + a^i_k a_n (a^i_k)^m u \\ &= 0, \end{aligned} \tag{3.3}$$

as required. In particular, we see that $a_n v = a_n(a^i_k)^r u = 0$. Therefore, $d(v) \leq d(u)$. Since $a^i_k v = 0$ and $a^i_k u \neq 0$, we have $d(v) < d(u)$.

Claim 2: *There is a nonzero $u \in M$ such that $\mathfrak{g}(n)u = 0$ for $n > 0$ and $\mathfrak{g}_+ u = 0$ where $\mathfrak{g}_+ = \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$ for a fixed positive root system Δ_+.*

Set

$$\Omega(M) = \{u \in M | (t\mathbb{C}[t] \otimes \mathfrak{g})u = 0\}. \tag{3.4}$$

Then $\Omega(M)$ is a nonzero $\mathfrak{g}$-submodule of M by Claim 1. Let $0 \neq e_\theta \in \mathfrak{g}_\theta$ where θ is the longest positive root in Δ. Then $Y_M(e_\theta, z)^{\ell+1} = 0$ (see [DL] and [FZ]). Extracting the coefficient of $z^{-\ell-1}$ from $Y_M(e_\theta, z)^{\ell+1}\Omega(M) = 0$, we obtain $e_\theta^{\ell+1}\Omega(M) = 0$. By Proposition 5.1.2 of [L1] $\Omega(M)$ is a direct sum of finite-dimensional irreducible $\mathfrak{g}$-modules. Then any highest weight vector for $\mathfrak{g}$ in $\Omega(M)$ meets our need.

Claim 3: *Any lowest weight vector for $\hat{\mathfrak{g}}$ in M generates a simple $L(\ell, 0)$-module.* Let u be a lowest weight vector for $\hat{\mathfrak{g}}$ in M. Extracting the constant term from $Y_M(e_\theta, z)^{\ell+1}u = 0$, we obtain $(e_\theta)_{-1}^{\ell+1}u = 0$. Then u generates an integrable highest weight $\hat{\mathfrak{g}}$-module. It follows from [K1] that u generates an irreducible $\hat{\mathfrak{g}}$-module of level ℓ. Since any submodule of M for the affine Lie algebra is a submodule of M for $L(\ell, 0)$, such u generates a simple $L(\ell, 0)$-module. $\square$

Remark 3.8 *This theorem has been proved in [DLM2] under the assumption that $t\mathbb{C}[t]\otimes\mathfrak{h}$ acts locally nilpotently on any weak module. See Proposition 5.6 in [DLM2].*

Remark 3.9 *Recall that a $\hat{\mathfrak{g}}$-module M is called* restricted *(cf. [K1]) if for any $u \in M$, there is an integer k such that $(t^n \otimes \mathfrak{g})u = 0$ for $n > k$; M is called an* integrable *module if the Chevalley generators e_i, f_i of $\hat{\mathfrak{g}}$ act locally finitely on M [K2] (note that in the definition of integrable module, we do not assume that the action of $\mathfrak{h}$ is semisimple). At affine Lie algebra level, Proposition 3.5 and Theorem 3.7 essentially assert that any restricted integrable $\hat{\mathfrak{g}}$-module is a direct sum of irreducible highest weight integrable $\hat{\mathfrak{g}}$-modules.*

Next we turn our attention to the vertex operator algebras $L(c_{p,q}, 0)$.

First we recall some results from [DL]. Let V be a vertex operator algebra and M be a weak V-module. Then for any $u, v \in V$ we have

$$Y(u_{-1}v, z) = Y(u, z)^- Y(v, z) + Y(v, z)Y(u, z)^+, \tag{3.5}$$

where

$$Y(u, z)^+ = \sum_{n\geq 0} u_n z^{-n-1},\ Y(u, z)^- = \sum_{n<0} u_n z^{-n-1}. \tag{3.6}$$

Note that $Y(u, z) = Y(u, z)^+ + Y(u, z)^-$ and that $Y(u, z)^+$ (reps. $Y(u, z)^-$) involves only nonpositive (resp. nonnegative) powers of z.

For convenience we will write $c = c_{p,q}$. For any nonnegative integer n, we set $\omega^{(n)} = \frac{1}{n!}L(-1)^n\omega$. Then $L(-n-2) = \omega_{-1}^{(n)}$. We need the following lemmas.

Lemma 3.10 *Let M be a weak $L(c,0)$-module and $u \in M$. Let k be a positive integer such that $L(k)u \neq 0$ and that $L(n)u = 0$ whenever $n > k$. Then for any nonnegative integers $n_1, ..., n_r$ the lowest power of z in $Y_M(\omega_{-1}^{(n_1)} \cdots \omega_{-1}^{(n_r)}\mathbf{1}, z)u$ (in the sense that the coefficients of z^m is zero whenever m is smaller than the lowest power) is $-r(k+2) - n_1 - \cdots - n_r$ with coefficient $\prod_{i=1}^{r} \binom{-k-2}{n_i} L(k)^r u$.*

Proof. We prove this lemma by induction on r. If $r = 1$, we have:

$$\begin{aligned} Y_M(\omega_{-1}^{(n_1)}\mathbf{1}, z)u &= Y_M(\omega^{(n_1)}, z)u \\ &= \frac{1}{n_1!}\left(\frac{d}{dz}\right)^{n_1} Y_M(\omega, z)u \\ &= \sum_{n\in\mathbb{Z}} \binom{-n-2}{n_1} z^{-n-2-n_1} L(n)u \\ &= \sum_{n\leq k} \binom{-n-2}{n_1} z^{-n-2-n_1} L(n)u. \end{aligned} \tag{3.7}$$

Then the lowest power of z is $-(k+2) - n_1$ with a coefficient $\binom{-k-2}{n_1}L(k)u$. That is, the lemma holds for $r = 1$.

Suppose that this lemma holds for some positive integer r. By formula (3.5) we have:

$$\begin{aligned} Y_M(\omega_{-1}^{(n_1)} \cdots \omega_{-1}^{(n_r)}\omega_{-1}^{(n_{r+1})}\mathbf{1}, z)u &= Y_M(\omega^{(n_1)}, z)^- Y_M(\omega_{-1}^{(n_2)} \cdots \omega_{-1}^{(n_{r+1})}\mathbf{1}, z)u \\ &+ Y_M(\omega_{-1}^{(n_2)} \cdots \omega_{-1}^{(n_{r+1})}\mathbf{1}, z) Y_M(\omega^{(n_1)}, z)^+ u. \end{aligned} \tag{3.8}$$

Since $Y_M(\omega^{(n_1)}, z)^-$ involves only nonnegative powers of z, it follows from the inductive assumption, the lowest power of z in the first term of the right hand side of (3.8) is $-r(k+2) - n_2 - \cdots - n_{r+1}$. It is easy to observe that for any v in the algebra,

$$Y_M(L(-1)v, z)^+ = \frac{d}{dz} Y_M(v, z)^+.$$

Thus

$$\begin{aligned} & Y_M(\omega_{-1}^{(n_2)} \cdots \omega_{-1}^{(n_{r+1})}\mathbf{1}, z) Y_M(\omega^{(n_1)}, z)^+ u \\ = & \frac{1}{n_1!} Y_M(\omega_{-1}^{(n_2)} \cdots \omega_{-1}^{(n_{r+1})}\mathbf{1}, z) \left(\frac{d}{dz}\right)^{n_1} Y_M(\omega, z)^+ u \\ = & \sum_{n=-1}^{k} \binom{-n-2}{n_1} Y_M(\omega_{-1}^{(n_2)} \cdots \omega_{-1}^{(n_{r+1})}\mathbf{1}, z) L(n)u z^{-n-2-n_1} \\ = & \sum_{n=0}^{k} \binom{-n-2}{n_1} Y_M(\omega_{-1}^{(n_2)} \cdots \omega_{-1}^{(n_{r+1})}\mathbf{1}, z) L(n)u z^{-n-2-n_1} \\ & + \binom{-1}{n_1} L(-1) Y_M(\omega_{-1}^{(n_2)} \cdots \omega_{-1}^{(n_{r+1})}\mathbf{1}, z) u z^{-1-n_1} \\ & + \binom{-1}{n_1} \left(\frac{d}{dz} Y_M(\omega_{-1}^{(n_2)} \cdots \omega_{-1}^{(n_{r+1})}\mathbf{1}, z)\right) u z^{-1-n_1}. \end{aligned}$$

Note that $L(m)L(n)u = (m-n)L(m+n)u + L(n)L(m)u = 0$ for $0 \leq n \leq k$ and $m > k$. Applying the inductive hypothesis to $L(n)u$ we see that the lowest power of z in

$$\sum_{n=0}^{k} \binom{-n-2}{n_1} Y_M(\omega_{-1}^{(n_2)} \cdots \omega_{-1}^{(n_{r+1})}\mathbf{1}, z)L(n)uz^{-n-2-n_1}$$

is $-(r+1)(k+2)-n_1-\cdots-n_{r+1}$ with coefficient $\prod_{i=1}^{r+1} \binom{-k-2}{n_i} L(k)^{r+1}u$. Also by the induction assumption the lowest power of z in $L(-1)Y_M(\omega_{-1}^{(n_2)} \cdots \omega_{-1}^{(n_{r+1})}\mathbf{1}, z)uz^{-1-n_1}$ is $-r(k+2)-n_1-\cdots-n_{r+1}-1$ and the lowest power of z in $\left(\frac{d}{dz}Y_M(\omega_{-1}^{(n_2)} \cdots \omega_{-1}^{(n_{r+1})}\mathbf{1}, z)\right)uz^{-1-n_1}$ is $-r(k+2)-n_1-\cdots-n_{r+1}-2$. Thus the lowest power of z in the second term of right hand side of (3.8) is $-(r+1)(k+2)-n_1-\cdots-n_{r+1}$ with coefficient $\prod_{i=1}^{r+1} \binom{-k-2}{n_i} L(k)^{r+1}u$, as desired. $\square$

Let V be a vertex operator algebra and let $A(V)$ (which is a certain quotient space of V modulo a subspace $O(V)$) be the corresponding associative algebra defined in [Z]. We refer the reader to [Z] for details. Recall from [DLM1] or [L2] that for any weak V-module M, $\Omega(M)$ consists of vectors $u \in M$ such that $a_m u = 0$ for any homogeneous element $a \in V$ and for any $m > \operatorname{wt} a - 1$. In other words, $u \in \Omega(M)$ if and only if $z^m Y_M(a, z)u \in M[[z]]$ for any homogeneous element $a \in V$ and for any $m > \operatorname{wt} a - 1$. The following result can be found in [DLM1], [L2] and [Z].

Lemma 3.11 *(1) $\omega + O(V)$ is in the center of $A(V)$.*

(2) $A(V)$ is semisimple if V is rational.

(3) $\Omega(M)$ is an $A(V)$-module under the action $a + O(V) \mapsto a_{\operatorname{wt} a-1}$ for homogeneous $a \in V$.

Now we take $V = L(c, 0)$. Set $\bar{\Omega}(M) = \{u \in M | L(n)u = 0 \text{ for any } n > 0\}$. Then it is clear that $\Omega(M) \subseteq \bar{\Omega}(M)$.

Lemma 3.12 *Let M be a weak $L(c, 0)$-module. Then $\Omega(M) = \bar{\Omega}(M)$.*

Proof. It suffices to prove that $a_m u = 0$ for any $u \in \bar{\Omega}(M)$ and for any homogeneous element $a \in L(c, 0)$ whenever $m > \operatorname{wt} a - 1$. We shall prove this by induction on the weight of a. If $\operatorname{wt} a = 0$, $a = \mathbf{1}$. Since $\mathbf{1}_m = 0$ for $m \geq 0$, there is nothing to prove. Suppose that $a_m u = 0$ for any homogeneous element $a \in L(c, 0)$ of weight less than n and for any $m > \operatorname{wt} a - 1$. Let $b \in L(c, 0)$ be a homogeneous element of weight n and let $m \in \mathbb{Z}$ such that $m > \operatorname{wt} b - 1$.

Let $a \in L(c, 0)$ be any homogeneous element of weight less than n, let k be any positive integer and let $m > \operatorname{wt}(L(-k)a) - 1$ $(= \operatorname{wt} a + k - 1)$. Then from the Jacobi identity

(2.4) we have:

$$
\begin{aligned}
&(L(-k)a)_m u = \mathrm{Res}_{z_0}\mathrm{Res}_{z_2} z_0^{1-k} z_2^m Y_M(Y(\omega, z_0)a, z_2)u \\
&\ = \mathrm{Res}_{z_1}\mathrm{Res}_{z_0}\mathrm{Res}_{z_2} z_0^{1-k} z_2^m \cdot \\
&\ \cdot \left(z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(\omega, z_1)Y_M(a, z_2)u - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(a, z_2)Y_M(\omega, z_1)u \right) \\
&\ = \mathrm{Res}_{z_1}\mathrm{Res}_{z_2}(z_1 - z_2)^{1-k} z_2^m Y_M(\omega, z_1)Y_M(a, z_2)u \\
&\ -\mathrm{Res}_{z_1}\mathrm{Res}_{z_2}(-z_2 + z_1)^{1-k} z_2^m Y_M(a, z_2)Y_M(\omega, z_1)u.
\end{aligned} \tag{3.9}
$$

Since $m > \mathrm{wt}\,(L(-k)a) - 1 = \mathrm{wt}a + k - 1 > \mathrm{wt}a - 1$, we have:

$$
\mathrm{Res}_{z_1}\mathrm{Res}_{z_2}(z_1 - z_2)^{1-k} z_2^m Y_M(\omega, z_1)Y_M(a, z_2)u = 0.
$$

For the second term, we have:

$$
\begin{aligned}
&\quad -\mathrm{Res}_{z_1}\mathrm{Res}_{z_2}(-z_2 + z_1)^{1-k} z_2^m Y_M(a, z_2)Y_M(\omega, z_1)u \\
&= -\mathrm{Res}_{z_2}(-1)^{1-k} z_2^{m+1-k} Y_M(a, z_2)L(-1)u - \mathrm{Res}_{z_2}(-1)^{-k}(1-k) z_2^{m-k} Y_M(a, z_2)L(0)u \\
&= -\mathrm{Res}_{z_2}(-1)^{1-k} z_2^{m+1-k} L(-1)Y_M(a, z_2)u + \mathrm{Res}_{z_2}(-1)^{1-k} z_2^{m+1-k} \frac{d}{dz_2} Y_M(a, z_2)u \\
&\quad -\mathrm{Res}_{z_2}(-1)^{-k}(1-k) z_2^{m-k} Y_M(a, z_2)L(0)u \\
&= -\mathrm{Res}_{z_2}(-1)^{1-k} z_2^{m+1-k} L(-1)Y_M(a, z_2)u - \mathrm{Res}_{z_2}(-1)^{1-k}(m+1-k) z_2^{m-k} Y_M(a, z_2)u \\
&\quad -\mathrm{Res}_{z_2}(-1)^{-k}(1-k) z_2^{m-k} Y_M(a, z_2)L(0)u \\
&= (-1)^k L(-1)a_{m+1-k}u + (-1)(m+1-k)a_{m-k}u + (-1)^{k-1}a_{m-k}L(0)u.
\end{aligned} \tag{3.10}
$$

Since $L(0)u \in \bar{\Omega}(M)$ and $m - k > \mathrm{wt}a - 1$ all the three terms in (3.10) are zero by the inductive hypothesis. Thus $(L(-k)a)_m u = 0$. Note that b is a linear combination of all $L(-k)a$, where $\mathrm{wt}a < n$ and k is a positive integer. This shows $b_m\bar{\Omega}(M) = 0$ for $m > \mathrm{wt}\,b - 1$, as desired. $\square$

Now we are in a position to prove

Theorem 3.13 *The vertex operator algebra $L(c, 0)$ associated with the lowest weight irreducible module for the Virasoro algebra with central charge $c = c_{p,q}$ is regular.*

Proof: By Proposition 3.5, it is enough to prove that any nonzero weak $L(c, 0)$-module M contains a simple $L(c, 0)$-module.

Claim 1: *The space $\Omega(M)$ is not zero.* For any $0 \neq u \in M$, we define $l(u)$ to be the integer k such that $L(k)u \neq 0$ and $L(n)u = 0$ whenever $n > k$. Since $L(n)u \neq 0$ for some n (because $c \neq 0$), $l(u)$ is well-defined. Suppose that $\Omega(M) = 0$. Then by Lemma 3.12 $l(u) \geq 1$ for any $0 \neq u \in M$. Let $0 \neq u \in M$ such that $l(u) = k$ is minimal. It is well known [FF] that there are two singular vectors in the Verma module $M(c, 0)$ for the Virasoro algebra. One singular vector is $L(-1)\mathbf{1}$ and the other is:

$$
v = L(-2)^{pq}\mathbf{1} + \sum a_{n_1,\cdots,n_r}\omega_{-1}^{(n_1)}\cdots\omega_{-1}^{(n_r)}\mathbf{1}, \tag{3.11}
$$

where the sum is over some $(n_1, \cdots, n_r) \in \mathbb{Z}_+^r$ such that $2pq = 2r + n_1 + \cdots + n_r$ and $n_1 + \cdots + n_r \neq 0$. By Lemma 3.10, the lowest power of z in

$$Y_M(L(-2)^{pq}\mathbf{1}, z)u = Y_M((\omega_{-1})^m\mathbf{1}, z)u$$

is $-pq(k+2)$ with $L(k)^{pq}u$ as its coefficient and the lowest power of z in

$$Y_M(\omega_{-1}^{(n_1)} \cdots \omega_{-1}^{(n_r)}\mathbf{1}, z)u$$

is greater than $-pq(k+2)$ for any nonnegative integers $n_1, \cdots, n_r$ such that $2pq = 2r + n_1 + \cdots + n_r$ and $n_1 + \cdots + n_r \neq 0$. Thus the coefficient of $z^{-pq(k+2)}$ in $Y_M(v, z)u$ is $L(k)^{pq}u$. Since $v = 0$ in $L(c, 0)$ we have $Y_M(v, z) = 0$. In particular the coefficient $L(k)^{pq}u$ of $z^{-pq(k+2)}$ in $Y_M(v, z)$ is zero. Let s be the nonnegative integer such that $L(k)^s u \neq 0$ and $L(k)^{s+1}u = 0$ and set $u' = L(k)^s u$. Then it is clear that $l(u') < l(u)$. This is a contradiction.

Claim 2: *Any weak $L(c, 0)$-module M contains a simple ordinary $L(c, 0)$-module.* Since $L(c, 0)$ is rational, Lemma 3.11 tells us that $A(L(c, 0))$ is semisimple and that the central element $\omega + O(L(c, 0))$ acts semisimply on $\Omega(M)$ as $L(0)$. Since $\Omega(M)$ is nonzero by Claim 1 we can take $0 \neq u \in \Omega(M)$ such that $L(0)u = hu$ where $h \in \mathbb{C}$. Again since $L(c, 0)$ is rational [W], u generates a simple (ordinary) $L(c, 0)$-module. The proof is complete. □

Corollary 3.14 *The moonshine module vertex operator algebra $V^\natural$ is regular.*

Proof. From [DMZ], $V^\natural$ contains $L(\frac{1}{2}, 0)^{\otimes 48}$ as a vertex operator subalgebra. Then the result follows from Theorem 3.13, Propositions 3.3 and 3.4. □

Finally we discuss the complete reducibility of weak V_L-modules for an even lattice L. We refer the reader to [FLM] and [D1] for the construction of V_L and related notations.

Let M be any weak V_L-module. Define the vacuum space

$$\Omega_M = \{u \in M | \alpha(i)u = 0 \text{ for } \alpha \in L,\ i > 0\}.$$

Lemma 3.15 *Let L be an even lattice. Then for any weak V_L-module M, $\Omega_M \neq 0$.*

Proof. For $u \in M$ then $A_u = \mathrm{span}\{\alpha(n)u | \alpha \in L, n > 0\}$ is finite-dimensional as $\alpha(n)u = 0$ if n is sufficiently large and as the rank of L is finite. Set $d(u) = \dim A_u$. Note that $d(u) = 0$ if and only if $u \in \Omega_M$. So it is enough to show that $d(u) = 0$ for some nonzero $u \in M$. Assume this is false, and take $0 \neq u \in M$ such that $d(u)$ is minimal.

Let k be the smallest positive integer such that $\alpha(k)u \neq 0$ and $\beta(n)u = 0$ whenever $n > k$ for some $\alpha \in L$ and all $\beta \in L$.

Let $a \in \hat{L}$ such that $\bar{a} = \alpha$. Then from the formula (3.4) of [D1] we have

$$\frac{d}{dz}Y(\iota(a), z) = Y(L(-1)\iota(a), z) = Y(\alpha(-1)\iota(a), z) = \alpha(z)^- Y(\iota(a), z) + Y(\iota(a), z)\alpha(z)^+$$

where

$$\alpha(z)^- = \sum_{n<0} \alpha(n) z^{-n-1}, \quad \alpha(z)^+ = \sum_{n\geq 0} \alpha(n) z^{-n-1}.$$

Clearly the submodule generated by u is not zero. Note that the vertex algebra V_L is simple (see [D1]). By Proposition 11.9 of [DL], $Y(\iota(a), z)u \neq 0$. Let r be an integer such that $\iota(a)_{r+m}u = 0$ and $\iota(a)_r u \neq 0$ for any positive integer m. Thus the lowest power of z in

$$\frac{d}{dz} Y(\iota(a), z)u = -\sum_{m\leq r} (m+1)\iota(a)_m u z^{-m-2}$$

is at most $-r-2$. It is obvious that the lowest power of z in $\alpha(z)^- Y(\iota(a), z)u$ is at most $-r-1$.

Use the following commutator formula which is a result from the Jacobi identity

$$[\beta(m), \iota(a)_n] = \langle \alpha, \beta \rangle \iota(a)_{m+n} \tag{3.12}$$

to obtain

$$\iota(a)_m \alpha(n) u = -\langle \alpha, \alpha \rangle \iota(a)_{m+n} u + \alpha(n)\iota(a)_m u = 0$$

if $m > r$ and $n \geq 0$. This gives

$$Y(\iota(a), z)\alpha(z)^+ u = \sum_{m\leq r} \sum_{n=0}^{k} \iota(a)_m \alpha(n) z^{-m-n-2}.$$

Thus the coefficient $\iota(a)_r \alpha(k) u$ of z^{-r-k-2} in the formula above is zero as k is positive. This shows by (3.12) again that $\alpha(n)\iota(a)_r u = 0$ for any positive integer greater than or equal to k.

Note from (3.12) that if $\beta(m)u = 0$ for positive m then $\beta(m)\iota(a)_r u = 0$. Thus $d(\iota(a)_r u) < d(u)$. This is a contradiction. $\square$

Theorem 3.16 *Let L be an even lattice. Then any weak V_L-module is completely reducible and any simple weak V_L-module is isomorphic to $V_{L+\beta}$ for some β in the dual lattice of L. In particular, V_L is regular if L is positive definite.*

Proof. By Lemma 3.15, $\Omega_M \neq 0$ for a weak V_L-module M. It is proved in [D1] that if M is also simple then it is necessarily isomorphic to $V_{L+\beta}$ for some β in the dual lattice of L. So it remains to show the complete reducibility of any weak V_L-module M.

Let W be the sum of all simple submodules of M. Assume that $M' = M/W$ is not zero. Then $\Omega_{M'} \neq 0$. It is essentially proved in [D1] that M' contains a simple module W^1/W (here W^1 is a weak V_L-submodule of M which contains W) generated by $w^1 + W$ where w^1 is a common eigenvector for the operators $\alpha(0)$ for $\alpha \in L$. It follows from the proof of Theorem 3.1 of [D1] that the submodule of M generated by w^1 is simple. Thus W^1 is a sum of certain simple submodules of V_L. This is a contradiction. $\square$

At this point we have proved that almost all known rational vertex operator algebras are regular. We conclude this paper by presenting the following conjecture:

Conjecture 3.17 *Any rational vertex operator algebra is regular.*

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